

# Constrained Paracomplex Mechanical Systems

Mehmet Tekkoyun \*

Department of Mathematics, Pamukkale University,

20070 Denizli, Turkey

Murat Sari †

Department of Mathematics, Pamukkale University,

20070 Denizli, Turkey

February 24, 2009

## Abstract

In this study, it is introduced paracomplex analogue of Lagrangians and Hamiltonians with constraints in the framework of para-Kählerian manifolds. The geometrical and mechanical results on the constrained mechanical system have also been discussed.

**Keywords:** Paracomplex geometry; Para-Hermitian and para-Kählerian manifolds; Lagrangian and Hamiltonian systems; Constraints.

**MSC:** 53C15, 70H03, 70H05.

## 1 Introduction

It is well known that the dynamics of Lagrangian and Hamiltonian formalisms is characterized by a suitable vector field defined on the tangent and cotangent bundles which are phase-spaces of velocities and momentum of a given configuration manifold. If  $Q$  is an  $m$ -dimensional configuration manifold and  $L : TQ \rightarrow \mathbf{R}$  is a regular Lagrangian function, then there is a unique vector field  $\xi_L$  on  $TQ$  such that dynamical equations

$$i_{\xi_L} \Phi_L = dE_L, \quad (1)$$

---

\*tekkoyun@pau.edu.tr

†msari@pau.edu.tr

where  $\Phi_L$  is the symplectic form and  $E_L$  is the energy associated to  $L$ . The Euler-Lagrange vector field  $\xi_L$  is a second order differential equation on  $Q$  since its integral curves are the solutions of the Euler-Lagrange equations given by

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \quad (2)$$

where  $q^i$  and  $(q^i, \dot{q}^i)$ ,  $1 \leq i \leq m$ , are coordinate systems of  $Q$  and  $TQ$ , respectively. The triple  $(TQ, \Phi_L, E_L)$  is called *Lagrangian mechanical system* on the tangent bundle  $TQ$ . Assume that  $(TQ, \Phi_L)$  is symplectic manifold and  $\bar{\omega} = \{\omega_1, \dots, \omega_r\}$  is a system of constraints on  $TQ$ . It is called as a *constraint* on  $TQ$  to a non-zero 1-form  $\omega = \wedge^a \omega_a$  on  $TQ$ , such that  $\wedge^a$ ,  $1 \leq a \leq r$ , are Lagrange multipliers. The quartet  $(TQ, \Phi_L, E_L, \bar{\omega})$  is said to be a *regular Lagrangian system with constraints*. The constraints  $\bar{\omega}$  are said to be classical constraints if the 1-forms  $\omega_a$  are basic. Then holonomic classical constraints define foliations on the configuration manifold  $Q$ , but holonomic constraints also admit foliations on the phase space of velocities  $TQ$ . As is the case in real studies, generally, a curve  $\alpha$  satisfying the Euler Lagrange equations for Lagrangian energy  $E_L$  does not satisfy the constraints. In order to satisfy the constraints, some additional forces act on the system as well as *force*  $dE_L$  for a curve  $\alpha$ . It is said that the quartet  $(TQ, \Phi_L, E_L, \bar{\omega})$  defines a *mechanical system with constraints* if vector field  $\xi$  given by the equations of motion

$$i_\xi \Phi_L = dE_L + \wedge^a \omega_a, \quad \omega_a(\xi) = 0, \quad (3)$$

is a second order differential equation. Then, it is given Euler-Lagrange equations with constraints as follows:

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = \wedge^a (\omega_a)_i, \quad (4)$$

If  $H : T^*Q \rightarrow \mathbf{R}$  is a regular Hamiltonian function then there is a unique vector field  $Z_H$  on cotangent bundle  $T^*Q$  such that dynamical equations

$$i_{Z_H} \Phi = dH, \quad (5)$$

where  $\Phi$  is the symplectic form and  $H$  stands for Hamiltonian function. The paths of the Hamiltonian vector field  $Z_H$  are the solutions of the Hamiltonian equations shown by

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad (6)$$

where  $q^i$  and  $(q^i, p_i)$ ,  $1 \leq i \leq m$ , are coordinates of  $Q$  and  $T^*Q$ . The triple  $(T^*Q, \Phi, H)$ , is called *Hamiltonian system* on the cotangent bundle  $T^*Q$  with symplectic form  $\Phi$ . Let  $T^*Q$  be symplectic manifold with closed symplectic form  $\Phi$ . Similar to constraints on  $TQ$ , by a

*constraint* on  $T^*Q$  is said to be a *non-zero 1-form*  $\omega = \wedge^a \omega_a$  on  $T^*Q$ . A set  $\overline{\omega} = \{\omega_1, \dots, \omega_s\}$  of  $s$  linearly independent 1-forms on  $T^*Q$  may be named to be a *system of constraints* on  $T^*Q$ . We say that a curve  $\alpha$  in  $T^*Q$  satisfies the constraints if  $\omega_a(\dot{\alpha}(t)) = 0$ ,  $1 \leq a \leq s$ .

Let  $(T^*Q, \Phi, H)$  be a Hamiltonian system on symplectic manifold  $T^*Q$  with closed symplectic form  $\Phi$ . Let us consider a Hamiltonian system  $(T^*Q, \Phi, H)$  together with a system  $\overline{\omega}$  of constraints on  $T^*Q$ . So, it is called  $(T^*Q, \Phi, H, \overline{\omega})$  to be a *Hamiltonian system with constraints*. In general, a curve  $\alpha$  satisfying the Hamiltonian equations for energy  $H$  does not satisfy the constraints. For a curve  $\alpha$  satisfying the constraints, some additional forces must act on the system in addition to the *force*  $dH$ . So, the dynamical equations of motion become

$$i_Z \Phi = dH + \wedge^a \omega_a, \quad \omega_a(Z) = 0, \quad (7)$$

where  $Z$  is a vector field on  $T^*Q$ . From (7), Hamiltonian equations with constraints is given by:

$$\begin{aligned} \frac{dq^i}{dt} &= \left( \frac{\partial H}{\partial p_i} + \wedge^a (B_a)_i \right), \\ \frac{dp_i}{dt} &= - \left( \frac{\partial H}{\partial q_i} + \wedge^a (A_a)_i \right), \\ (A_a)_i \frac{dq^i}{dt} + (B_a)_i \frac{dp_i}{dt} &= 0, \end{aligned} \quad (8)$$

where  $1 \leq i \leq m$ ,  $1 \leq a \leq s$ .

It can be easily understood that the above approach provides a good framework for studying Lagrangian and Hamiltonian formalisms of classical mechanics. There are some articles in [1, 2, 3, 4] and books in [5, 6] on differential geometric methods in mechanics. It is well known that (para)Kählerian manifolds play an essential role in various areas of mathematics and mathematical physics, in particular, in the theory of dynamical systems, algebraic geometry, the geometry of Einstein manifolds, quantum mechanics, quantum field theory, and in the theory of superstrings and nonlinear sigma-models, too. For example, it was shown in [7] that the reflector space of an Einstein self-dual non-Ricci flat 4-manifold as well as the reflector space of a paraquaternionic Kählerian manifold admit both Nearly para-Kählerian and almost para-Kählerian structures. Wade [8] showed that generalized paracomplex structures are in one-to-one correspondence with pairs of transversal Dirac structures on a smooth manifold. In [9], it was given a representation of the quadratic Dirac equation and the Maxwell equations in terms of the three-dimensional universal complex Clifford algebra  $C_{3,0}$ . Baylis and Jones introduced in [10] that a  $R_{3,0}$  Clifford algebra has enough structure to describe relativity as well as the more usual  $R_{1,3}$  Dirac algebra or the  $R_{3,1}$  Majorana algebra. In [11], Baylis represented relativistic space-time points as paravectors and applies these paravectors to electrodynamics. Tekkoyun [12] generalized the concept of Hamiltonian dynamics with constraints to complex case. In the

above studies; although paracomplex geometry, complex mechanical systems with constraints, Lagrangian and Hamiltonian mechanics were given in a tidy and nice way, they have not dealt with constrained paracomplex mechanical systems. Therefore, in this paper, as a contribution to the modern development of Lagrangian and Hamiltonian systems of classical mechanics, it was obtained paracomplex analogous of some topics in the geometric theory of constraints given in [3, 6, 12], and it has an important role in mechanical systems as pointed out in the above.

The present paper is structured as follows. In sections 1, 2 and 3, it is recalled paracomplex, para-Hermitian and para-Kählerian manifolds, and also para-Euler-Lagrange equations and para-Hamiltonian equations on para-Kählerian manifolds. In sections 4 and 5, paracomplex Euler-Lagrange and Hamiltonian equations with constraints on para-Kählerian manifold are deduced. In the conclusion section, the geometrical and mechanical theory of para mechanical system with constraints was presented.

## 2 Preliminaries

In this paper, all the geometrical objects are differentiable and the Einstein summation convention is in use. So,  $\mathbf{A}$ ,  $\mathcal{F}(TM)$ ,  $\chi(TM)$  and  $\Lambda^1(TM)$  denote the set of paracomplex numbers, the set of paracomplex functions on tangent bundle  $TM$ , the set of paracomplex vector fields on tangent bundle  $TM$  and the set of paracomplex 1-forms on tangent bundle  $TM$ , respectively. Here  $1 \leq i \leq m$ . Some geometric structures on the differential manifold  $M$  given by [13] can be extended to  $TM$  as follows:

### 2.1 Paracomplex Geometry

An *almost product structure*  $J$  on a tangent bundle  $TM$  of  $m$ -real dimensional configuration manifold  $M$  is a (1,1) tensor field  $J$  on  $TM$  such that  $J^2 = I$ . Here, the pair  $(TM, J)$  is called an *almost product manifold*. An *almost paracomplex manifold* is an almost product manifold  $(TM, J)$  such that the two eigenbundles  $TT^+M$  and  $TT^-M$  associated to the eigenvalues  $+1$  and  $-1$  of  $J$ , respectively, have the same rank. The dimension of an almost paracomplex manifold is necessarily even. Equivalently, a splitting of the tangent bundle  $TTM$  of tangent bundle  $TM$ , into the Whitney sum of two subbundles on  $TT^\pm M$  of the same fiber dimension is called an *almost paracomplex structure* on  $TM$ . From physical point of view, this splitting means that a reference frame has been chosen. Obviously, such a splitting is broken under reference frame transformations. An almost paracomplex structure on a  $2m$ -dimensional manifold  $TM$  may alternatively be defined as a  $G$ -structure on  $TM$  with structural group  $GL(n, \mathbf{R}) \times GL(n, \mathbf{R})$ .

A *paracomplex manifold* is an almost paracomplex manifold  $(TM, J)$  such that  $G$ -structure defined by tensor field  $J$  is integrable. Let  $(x^i)$  and  $(x^i, y^i)$  be a real coordinate system of  $M$  and  $TM$ , and  $\{(\frac{\partial}{\partial x^i})_p, (\frac{\partial}{\partial y^i})_p\}$  and  $\{(dx^i)_p, (dy^i)_p\}$  natural bases over  $\mathbf{R}$  of tangent space  $T_p(TM)$  and cotangent space  $T_p^*(TM)$  of  $TM$ , respectively. Then,  $J$  can be denoted as

$$J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}, J(\frac{\partial}{\partial y^i}) = \frac{\partial}{\partial x^i}. \quad (9)$$

Let  $z^i = x^i + \mathbf{j} y^i$ ,  $\mathbf{j}^2 = -1$ , be a paracomplex local coordinate system of  $TM$ . The vector and covector fields are defined, respectively, as follows:

$$(\frac{\partial}{\partial z^i})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^i})_p - \mathbf{j}(\frac{\partial}{\partial y^i})_p\}, (\frac{\partial}{\partial \bar{z}^i})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^i})_p + \mathbf{j}(\frac{\partial}{\partial y^i})_p\}, \quad (10)$$

$$(dz^i)_p = (dx^i)_p + \mathbf{j}(dy^i)_p, (d\bar{z}^i)_p = (dx^i)_p - \mathbf{j}(dy^i)_p. \quad (11)$$

The above equations represent the bases of tangent space  $T_p(TM)$  and cotangent space  $T_p^*(TM)$  of  $TM$ , respectively. Then the following results can be easily obtained, respectively:

$$J(\frac{\partial}{\partial z^i}) = -\mathbf{j}\frac{\partial}{\partial \bar{z}^i}, J(\frac{\partial}{\partial \bar{z}^i}) = \mathbf{j}\frac{\partial}{\partial z^i}, \quad (12)$$

$$J^*(dz^i) = -\mathbf{j}d\bar{z}^i, J^*(d\bar{z}^i) = \mathbf{j}dz^i. \quad (13)$$

Here,  $J^*$  stands for the dual endomorphism of cotangent space  $T_p^*(TM)$  of manifold  $TM$  satisfying  $J^{*2} = I$ .

An *almost para-Hermitian manifold*  $(TM, g, J)$  is a differentiable manifold  $TM$  endowed with an almost product structure  $J$  and a pseudo-Riemannian metric  $g$ , compatible in the sense that

$$g(JX, Y) + g(X, JY) = 0, \quad \forall X, Y \in \chi(TM). \quad (14)$$

An *almost para-Hermitian structure* on a differentiable manifold  $TM$  is  $G$ -structure on  $TM$  whose structural group is the representation of the paraunitary group  $U(n, \mathbf{A})$  given in [13]. A *para-Hermitian manifold* is a manifold with an integrable almost para-Hermitian structure  $(g, J)$ . 2-covariant skew tensor field  $\Phi$  defined by  $\Phi(X, Y) = g(X, JY)$  is so-called as *fundamental 2-form*. An almost para-Hermitian manifold  $(TM, g, J)$ , such that  $\Phi$  is closed, is so-called as an *almost para-Kählerian manifold*.

A para-Hermitian manifold  $(TM, g, J)$  is said to be a *para-Kählerian manifold* if  $\Phi$  is closed. Also, by means of geometric structures, one may show that  $(T^*M, g, J)$  is a *para-Kählerian manifold*.

## 2.2 Paracomplex Lagrangian Systems

In this section, some paracomplex fundamental concepts and para-Euler-Lagrange equations for classical mechanics structured on para-Kählerian manifold  $TM$  introduced in [4] can be recalled.

Let  $J$  be an almost paracomplex structure on the para-Kählerian manifold and  $(z^i, \bar{z}^i)$  its coordinates. Let a second order differential equation be vector field  $\xi_L$  given by:

$$\xi_L = \xi^i \frac{\partial}{\partial z^i} + \bar{\xi}^i \frac{\partial}{\partial \bar{z}^i}, \quad (15)$$

Then vector field  $V = J\xi_L$  is called a *para-Liouville vector field* on the para-Kählerian manifold  $TM$ . The mappings given by  $T, P : TM \rightarrow \mathbf{A}$  such that  $T = \frac{1}{2}m_i(\dot{z}^i)^2$ ,  $P = m_i\mathbf{g}h$  can be called as *the kinetic energy* and *the potential energy of system*, respectively, where  $m_i$  is mass of a mechanical system,  $\mathbf{g}$  is the gravity and  $h$  is the distance of the mechanical system on the para-Kählerian manifold to the origin. Then we call map  $L : TM \rightarrow \mathbf{A}$  such that  $L = T - P$  as *para-Lagrangian function* and the function given by  $E_L = V(L) - L$  as *the para-energy function* associated with  $L$ .

The operator  $i_J$  induced by  $J$  and shown as

$$i_J\omega(Z_1, Z_2, \dots, Z_r) = \sum_{i=1}^r \omega(Z_1, \dots, JZ_i, \dots, Z_r) \quad (16)$$

is said to be *vertical derivation*, where  $\omega \in \wedge^r TM$ ,  $Z_i \in \chi(TM)$ . The *vertical differentiation*  $d_J$  is defined as follows:

$$d_J = [i_J, d] = i_J d - d i_J, \quad (17)$$

where  $d$  is the usual exterior derivation. For almost paracomplex structure  $J$  determined by (12), the closed para-Kählerian form is the closed 2-form given by  $\Phi_L = -dd_J L$  such that

$$d_J = -\mathbf{j} \frac{\partial}{\partial z^i} dz^i + \mathbf{j} \frac{\partial}{\partial \bar{z}^i} d\bar{z}^i : \mathcal{F}(TM) \rightarrow \wedge^1 TM. \quad (18)$$

*Paracomplex-Euler-Lagrange equations* on para-Kählerian manifold  $TM$  are shown by

$$\mathbf{j} \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial z^i} \right) + \frac{\partial L}{\partial z^i} = 0, \quad \mathbf{j} \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \bar{z}^i} \right) - \frac{\partial L}{\partial \bar{z}^i} = 0. \quad (19)$$

Thus, the triple  $(TM, \Phi_L, \xi)$  is called a *paracomplex-mechanical system*.

## 2.3 Paracomplex Hamiltonian Systems

Here, we consider paracomplex-Hamiltonian equations for classical mechanics structured on para-Kählerian manifold  $T^*M$  introduced in [4]. Let  $T^*M$  be any para-Kählerian manifold and

$(z_i, \bar{z}_i)$  its coordinates.  $\{\frac{\partial}{\partial z_i}|_p, \frac{\partial}{\partial \bar{z}_i}|_p\}$  and  $\{dz_i|_p, (d\bar{z}_i)|_p\}$  be bases over paracomplex number  $\mathbf{A}$  of tangent space  $T_p(TM)$  and cotangent space  $T_p^*(TM)$  of  $TM$ . Assume that  $J^*$  is an almost paracomplex structure given by  $J^*(dz_i) = -\mathbf{j}d\bar{z}_i$ ,  $J^*(d\bar{z}_i) = \mathbf{j}dz_i$  and  $\lambda$  is a para-Liouville form given by  $\lambda = J^*(\omega) = \frac{1}{2}\mathbf{j}(z_id\bar{z}_i - \bar{z}_idz_i)$  such that paracomplex 1-form  $\omega = \frac{1}{2}(z_id\bar{z}_i + \bar{z}_idz_i)$  on  $T^*M$ . If  $\Phi = -d\lambda$  is closed para-Kählerian form, then  $\Phi$  is also a para-symplectic structure on  $T^*M$ .

Let  $T^*M$  be para-Kählerian manifold with closed para-Kählerian form  $\Phi$ . Then para-Hamiltonian vector field  $Z_H$  on  $T^*M$  with closed form  $\Phi$  can be given by:

$$Z_H = -\mathbf{j}\frac{\partial H}{\partial \bar{z}_i}\frac{\partial}{\partial z_i} + \mathbf{j}\frac{\partial H}{\partial z_i}\frac{\partial}{\partial \bar{z}_i}. \quad (20)$$

According to (5), *para-Hamiltonian equations* on para-Kählerian manifold  $T^*M$  are denoted by equations of

$$\frac{dz_i}{dt} = -\mathbf{j}\frac{\partial H}{\partial \bar{z}_i}, \quad \frac{d\bar{z}_i}{dt} = \mathbf{j}\frac{\partial H}{\partial z_i}. \quad (21)$$

**Example:** A central force field  $f(\rho) = A\rho^{\alpha-1}$  ( $\alpha \neq 0, 1$ ) acts on a body with mass  $m$  in a constant gravitational field. Then let us find out the para-Lagrangian and para-Hamiltonian equations of the motion by assuming the body always on the vertical plane.

The para-Lagrangian and para-Hamiltonian functions of the system are, respectively,

$$L = \frac{1}{2}m \dot{z}\dot{\bar{z}} - \frac{A}{\alpha}(\sqrt{z\bar{z}})^\alpha - \mathbf{j}mg \frac{(z - \bar{z})\sqrt{z\bar{z}}}{(z + \bar{z})\sqrt{1 - \frac{(z - \bar{z})^2}{(z + \bar{z})^2}}},$$

$$H = \frac{1}{2}m \dot{z}\dot{\bar{z}} + \frac{A}{\alpha}(\sqrt{z\bar{z}})^\alpha + \mathbf{j}mg \frac{(z - \bar{z})\sqrt{z\bar{z}}}{(z + \bar{z})\sqrt{1 - \frac{(z - \bar{z})^2}{(z + \bar{z})^2}}}.$$

Then, using (19) and (21), the so-called para-Lagrangian and para-Hamiltonian equations of the motion on the para-mechanical systems, can be obtained, respectively, as follows:

$$L1 : \quad \mathbf{j}\frac{\partial}{\partial t}S - S = 0, \quad L2 : \quad \mathbf{j}\frac{\partial}{\partial t}U + U = 0,$$

such that

$$S = -\frac{A}{2z}(\sqrt{z\bar{z}})^\alpha - \mathbf{j}\frac{mg(z - \bar{z})\bar{z}}{2\sqrt{z\bar{z}}(z + \bar{z})W} - \mathbf{j}\frac{mg\sqrt{z\bar{z}}}{(z + \bar{z})W} \\ + \mathbf{j}\frac{mg\sqrt{z\bar{z}}(z - \bar{z})}{(z + \bar{z})^2W} + \mathbf{j}\frac{mg\sqrt{z\bar{z}}(z - \bar{z})(-\frac{(z - \bar{z})}{(z + \bar{z})^2} + \frac{(z - \bar{z})^2}{(z + \bar{z})^3})}{(z + \bar{z})W^3},$$

$$\begin{aligned}
U = & -\frac{A}{2\bar{z}}(\sqrt{z\bar{z}})^\alpha - \mathbf{j} \frac{mg(z-\bar{z})z}{2\sqrt{z\bar{z}}(z+\bar{z})W} + \mathbf{j} \frac{mg\sqrt{z\bar{z}}}{(z+\bar{z})W} \\
& + \mathbf{j} \frac{mg\sqrt{z\bar{z}}(z-\bar{z})}{(z+\bar{z})^2W} + \mathbf{j} \frac{mg\sqrt{z\bar{z}}(z-\bar{z})\left(\frac{(z-\bar{z})}{(z+\bar{z})^2} + \frac{(z-\bar{z})^2}{(z+\bar{z})^3}\right)}{(z+\bar{z})W^3}
\end{aligned}$$

and

$$\begin{aligned}
H1 : \quad \frac{dz}{dt} = & -\mathbf{j} \left( \frac{A}{2\bar{z}}(\sqrt{z\bar{z}})^\alpha + \mathbf{j} \frac{mg(z-\bar{z})z}{2\sqrt{z\bar{z}}(z+\bar{z})W} - \mathbf{j} \frac{mg\sqrt{z\bar{z}}}{(z+\bar{z})W} \right. \\
& \left. - \mathbf{j} \frac{mg\sqrt{z\bar{z}}(z-\bar{z})}{(z+\bar{z})^2W} - \mathbf{j} \frac{mg\sqrt{z\bar{z}}(z-\bar{z})\left(\frac{(z-\bar{z})}{(z+\bar{z})^2} + \frac{(z-\bar{z})^2}{(z+\bar{z})^3}\right)}{(z+\bar{z})W^3} \right), \\
H2 : \quad \frac{d\bar{z}}{dt} = & \mathbf{j} \left( \frac{A}{2z}(\sqrt{z\bar{z}})^\alpha + \mathbf{j} \frac{mg(z-\bar{z})\bar{z}}{2\sqrt{z\bar{z}}(z+\bar{z})W} + \mathbf{j} \frac{mg\sqrt{z\bar{z}}}{(z+\bar{z})W} \right. \\
& \left. - \mathbf{j} \frac{mg\sqrt{z\bar{z}}(z-\bar{z})}{(z+\bar{z})^2W} - \mathbf{j} \frac{mg\sqrt{z\bar{z}}(z-\bar{z})\left(-\frac{(z-\bar{z})}{(z+\bar{z})^2} + \frac{(z-\bar{z})^2}{(z+\bar{z})^3}\right)}{(z+\bar{z})W^3} \right).
\end{aligned}$$

where  $W = \sqrt{1 - \frac{(z-\bar{z})^2}{(z+\bar{z})^2}}$ .

### 3 Constrained Paracomplex Lagrangians

In this section, we obtain para-Euler-Lagrange equations with constraints for classical mechanics structured on para-Kählerian manifold  $TM$ .

Let  $J$  be an almost paracomplex structure on the para-Kählerian manifold and  $(z^i, \bar{z}^i)$  its coordinates. Let us take a second order differential equation to the vector field  $\xi$  given by:

$$\xi = \xi_L + \wedge^a \omega_a = \xi^i \frac{\partial}{\partial z^i} + \bar{\xi}^i \frac{\partial}{\partial \bar{z}^i} + \wedge^a \omega_a, \quad 1 \leq a \leq r, \quad (22)$$

The vector field  $V = J\xi_L$  calculated by

$$-\mathbf{j} \xi^i \frac{\partial}{\partial z^i} + \mathbf{j} \bar{\xi}^i \frac{\partial}{\partial \bar{z}^i}, \quad (23)$$

is *para-Liouville vector field* on the para-Kählerian manifold  $TM$ . The closed 2-form expressed by  $\Phi_L = -dd_J L$  is found to be:

$$\begin{aligned}
\Phi_L = & -\mathbf{j} \frac{\partial^2 L}{\partial z^j \partial \bar{z}^i} dz^j \wedge d\bar{z}^i + \mathbf{j} \frac{\partial^2 L}{\partial \bar{z}^j \partial z^i} d\bar{z}^j \wedge dz^i \\
& -\mathbf{j} \frac{\partial^2 L}{\partial z^j \partial \bar{z}^i} dz^j \wedge d\bar{z}^i - \mathbf{j} \frac{\partial^2 L}{\partial \bar{z}^j \partial z^i} d\bar{z}^j \wedge dz^i,
\end{aligned} \quad (24)$$



where

$$d_J = -\mathbf{j} \frac{\partial}{\partial z^i} dz^i + \mathbf{j} \frac{\partial}{\partial \bar{z}^i} d\bar{z}^i : \mathcal{F}(TM) \rightarrow \wedge^1 TM. \quad (25)$$

If  $\xi$  is a second order differential equation defined by (3), then we have

$$\begin{aligned} i_\xi \Phi_L = & -\mathbf{j} \xi^i \frac{\partial^2 L}{\partial z^j \partial z^i} \delta_i^j dz^i + \mathbf{j} \xi^i \frac{\partial^2 L}{\partial z^j \partial z^i} dz^j + \mathbf{j} \bar{\xi}^i \frac{\partial^2 L}{\partial \bar{z}^j \partial z^i} \delta_i^j dz^i - \mathbf{j} \xi^i \frac{\partial^2 L}{\partial \bar{z}^j \partial z^i} d\bar{z}^j \\ & -\mathbf{j} \xi^i \frac{\partial^2 L}{\partial z^j \partial \bar{z}^i} \delta_i^j d\bar{z}^i + \mathbf{j} \bar{\xi}^i \frac{\partial^2 L}{\partial z^j \partial \bar{z}^i} dz^j - \mathbf{j} \bar{\xi}^i \frac{\partial^2 L}{\partial \bar{z}^j \partial \bar{z}^i} \delta_i^j d\bar{z}^i + \mathbf{j} \bar{\xi}^i \frac{\partial^2 L}{\partial \bar{z}^j \partial \bar{z}^i} d\bar{z}^j. \end{aligned} \quad (26)$$

Since closed para-Kählerian form  $\Phi_L$  on  $TM$  is para-symplectic structure, it is obtained

$$E_L = -\mathbf{j} \xi^i \frac{\partial L}{\partial z^i} + \mathbf{j} \bar{\xi}^i \frac{\partial L}{\partial \bar{z}^i} - L \quad (27)$$

and hence

$$\begin{aligned} dE_L + \wedge^a \omega_a = & -\mathbf{j} \xi^i \frac{\partial^2 L}{\partial z^j \partial z^i} dz^j + \mathbf{j} \bar{\xi}^i \frac{\partial^2 L}{\partial z^j \partial \bar{z}^i} dz^j - \frac{\partial L}{\partial z^j} dz^j \\ & -\mathbf{j} \xi^i \frac{\partial^2 L}{\partial \bar{z}^j \partial z^i} d\bar{z}^j + \mathbf{j} \bar{\xi}^i \frac{\partial^2 L}{\partial \bar{z}^j \partial \bar{z}^i} d\bar{z}^j - \frac{\partial L}{\partial \bar{z}^j} d\bar{z}^j + \wedge^a \omega_a. \end{aligned} \quad (28)$$

According to (3), if (26) and (28) are equal to each other, then the following equation can be obtained:

$$\begin{aligned} & +\mathbf{j} \xi^i \frac{\partial^2 L}{\partial z^j \partial z^i} dz^j + \mathbf{j} \bar{\xi}^i \frac{\partial^2 L}{\partial \bar{z}^j \partial z^i} dz^j + \frac{\partial L}{\partial z^j} dz^j \\ & -\mathbf{j} \xi^i \frac{\partial^2 L}{\partial z^j \partial \bar{z}^i} d\bar{z}^j - \mathbf{j} \bar{\xi}^i \frac{\partial^2 L}{\partial \bar{z}^j \partial \bar{z}^i} d\bar{z}^j + \frac{\partial L}{\partial \bar{z}^j} d\bar{z}^j = \wedge^a \omega_a \end{aligned} \quad (29)$$

Now, let curve  $\alpha : \mathbf{A} \rightarrow TM$  be integral curve of  $\xi$ , which satisfies equations of

$$\begin{aligned} & +\mathbf{j} \left[ \xi^j \frac{\partial^2 L}{\partial z^j \partial z^i} + \dot{\xi}^i \frac{\partial^2 L}{\partial z^j \partial z^i} \right] dz^j + \frac{\partial L}{\partial z^j} dz^j \\ & -\mathbf{j} \left[ \xi^j \frac{\partial^2 L}{\partial z^j \partial \bar{z}^i} + \dot{\xi}^j \frac{\partial^2 L}{\partial z^j \partial \bar{z}^i} \right] d\bar{z}^j + \frac{\partial L}{\partial \bar{z}^j} d\bar{z}^j = \wedge^a \omega_a, \end{aligned} \quad (30)$$

where the dots mean derivatives with respect to time and  $\omega_a = (\omega_a)_i dz^i + (\dot{\omega}_a)_i d\bar{z}^i$ .

This refers to equations of

$$\frac{\partial L}{\partial z^i} + \mathbf{j} \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial z^i} \right) = \wedge^a (\omega_a)_i, \quad \frac{\partial L}{\partial \bar{z}^i} - \mathbf{j} \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \bar{z}^i} \right) = \wedge^a (\dot{\omega}_a)_i. \quad (31)$$

Thus, the equations obtained in (31) on para-Kählerian manifold  $TM$  are so-called as *constrained paracomplex Euler-Lagrange equations*. Then the quartet  $(TM, \Phi_L, \xi, \bar{\omega})$  is named *constrained paracomplex mechanical system*.

## 4 Constrained Paracomplex Hamiltonians

Here, we conclude paracomplex Hamiltonian equations with constraints on para-Kählerian manifold  $T^*M$ . Similar to (5), the vector fields on  $T^*M$  satisfying the condition

$$i_{Z_a}\Phi = \omega_a, \quad 1 \leq a \leq s, \quad (32)$$

can be represented by  $Z_a$ .

**Proposition:** Let  $T^*M$  be para-Kaehlerian manifold with closed para-Kählerian form  $\Phi$ . Let us consider vector field  $Z_a$  on  $T^*M$  given by:

$$Z_a = -\mathbf{j}(B_a)_i \frac{\partial}{\partial z_i} + \mathbf{j}(A_a)_i \frac{\partial}{\partial \bar{z}_i}, \quad (33)$$

**Proof:** Let  $T^*M$  be para-Kählerian manifold with form  $\Phi$ . Consider that vector field  $Z_a$  is given by

$$Z_a = (Z_a)_i \frac{\partial}{\partial z_i} + (\bar{Z}_a)_i \frac{\partial}{\partial \bar{z}_i}. \quad (34)$$

From (32),  $i_{Z_a}\Phi$  can be calculated as

$$i_{Z_a}(-d\lambda) = \mathbf{j}(\bar{Z}_a)_i dz_i - \mathbf{j}(Z_a)_i d\bar{z}_i. \quad (35)$$

Moreover, we set

$$\omega_a = (A_a)_i dz_i + (B_a)_i d\bar{z}_i \quad (36)$$

According to (32), if (35) and (36) are equal to each other, proof finishes.  $\diamond$

Now, with the case of (5) and (7) and (32); one may easily deduce

$$Z = Z_H + \wedge^a Z_a. \quad (37)$$

Hence, by means of (21), (33) and (37) we obtain the following vector field

$$Z = -\mathbf{j}\left(\frac{\partial H}{\partial \bar{z}_i} + \wedge^a (B_a)_i\right) \frac{\partial}{\partial z_i} + \mathbf{j}\left(\frac{\partial H}{\partial z_i} + \wedge^a (A_a)_i\right) \frac{\partial}{\partial \bar{z}_i}. \quad (38)$$

Suppose that curve

$$\alpha : I \subset \mathbf{A} \rightarrow T^*M$$

be an integral curve of paracomplex vector field  $Z$  given by (38), i.e.,

$$Z(\alpha(t)) = \dot{\alpha}(t), \quad t \in I. \quad (39)$$

In the local coordinates, for  $\alpha(t) = (z_i(t), \bar{z}_i(t))$ , we have

$$\dot{\alpha}(t) = \frac{dz_i}{dt} \frac{\partial}{\partial z_i} + \frac{d\bar{z}_i}{dt} \frac{\partial}{\partial \bar{z}_i}. \quad (40)$$

Then we reach the following equations

$$\begin{aligned}
\frac{dz_i}{dt} &= -\mathbf{j}(\frac{\partial H}{\partial \bar{z}_i} + \wedge^a(B_a)_i), \\
\frac{d\bar{z}_i}{dt} &= \mathbf{j}(\frac{\partial H}{\partial z_i} + \wedge^a(A_a)_i), \\
(A_a)_i \frac{dz_i}{dt} + (B_a)_i \frac{d\bar{z}_i}{dt} &= 0,
\end{aligned} \tag{41}$$

which are so-called as *constrained paracomplex Hamiltonian equations* on para-Kählerian manifold  $T^*M$ . Here  $1 \leq a \leq s$ . Then the quartet  $(T^*M, \Phi, H, \bar{\omega})$  is named *constrained paracomplex mechanical system*.

## 5 Conclusion

Finally, considering the above, complex analogous of the geometrical and mechanical meaning of constraints given in [3, 6, 12, 14] can be explained as follows:

1) Let  $\bar{\omega}$  be a system of constraints on para-Kählerian manifold  $TM$  or  $T^*M$ . Then it may be defined a distribution  $D$  or  $D^*$  on  $\bar{\omega}$  as follows:

$$\begin{aligned}
D(x) &= \{\xi \in T_x TM \mid \omega_a(\xi) = 0, \text{ for all } a, 1 \leq a \leq r\} \\
D^*(x) &= \{Z \in T_x T^*M \mid \omega_a(Z) = 0, \text{ for all } a, 1 \leq a \leq s\}
\end{aligned} \tag{42}$$

Thus  $D$  or  $D^*$  is  $(2m - r)$  or  $(2m - s)$ -dimensional distribution on  $TM$  or  $T^*M$ . In this case, a system of paracomplex constraints  $\bar{\omega}$  is paraholonomic, if the distribution  $D$  or  $D^*$  is integrable; otherwise  $\bar{\omega}$  is paraanholonomic. Hence,  $\bar{\omega}$  is paraholonomic if and only if the ideal  $\rho$  of  $\wedge TM$  or  $\wedge T^*M$  generated by  $\bar{\omega}$  is a differential ideal, i.e.,  $d\rho \subset \rho$ . Obviously, (31) and (41) hold both paraholonomic and paraanholonomic constraints. The motion for a system of paraholonomic constraints lies on a specific leaf of the foliation defined by  $D$  or  $D^*$ .

2) From (3) and (7), the following equalities can be obtained:

$$\begin{aligned}
0 &= (i_\xi \Phi)(\xi) = dE_L(\xi) = \xi(E_L), \\
0 &= (i_Z \omega)(Z) = dH(Z) = Z(H).
\end{aligned} \tag{43}$$

So, Lagrangian energy  $E_L$  and Hamiltonian energy  $H$  of (31) and (41) for a solution  $\alpha(t)$  are, respectively, conserved.

### Acknowledgment

We are very grateful to Professor M. Adak for his valuable comments, remarks and suggestions.

## References

- [1] M. Crampin, On the Differential Geometry of Euler-Lagrange Equations, and the Inverse Problem of Lagrangian Dynamics, J. Phys. A: Math. Gen. 14 (1981), pp. 2567-2575.
- [2] N. Nutku, Hamiltonian Formulation of KdV Equation, J. Math. Phys. 25 (1984), pp. 2007-2008.
- [3] W. R. Weber, Hamiltonian Systems with Constraints and their Meaning in Mechanics, Arc. Rat. Mech. Anal., 91(1985), pp. 309-335.
- [4] M. Tekkoyun, On Para- Euler- Lagrange and Para-Hamiltonian Equations, Physics Letters A, 340 (2005), pp. 7-11.
- [5] M. De Leon, P.R. Rodrigues, Generalized Classical Mechanics and Fields Theory, North-Holland Math. St.,112, Elsevier Sc. Pub. Com. Inc., Amsterdam, 1985.
- [6] M. De Leon, P.R. Rodrigues, Methods of Differential Geometry in Analytical Mechanics, North-Holland Math. St.,152, Elsevier Sc. Pub. Com. Inc., Amsterdam, 1989.
- [7] S. Ivanov, S. Zamkovoy, ParaHermitian and Paraquaternionic Manifolds, Differential Geometry and its Applications, 23 (2005), pp. 205-234
- [8] A. Wade, Dirac structures and paracomplex manifolds, Differential Geometry, C. R. Acad. Sci. Paris, Ser. I , 338 (2004), pp. 889-894.
- [9] S. Ulrych, Relativistic quantum physics with hyperbolic numbers, Physics Letters B, 625 (2005), pp. 313-323.
- [10] W.E. Baylis, G. Jones, J. Phys. A (Math Gen) 22 (1989), pp. 1-16.
- [11] W.E. Baylis, Electrodynamics: A Modern Geometrical Approach, Birkhauser, Boston, 1999.
- [12] M. Tekkoyun, A Note On Constrained Complex Hamiltonian Mechanics, Differential Geometry-Dynamical Systems (DGDS), 8 (2006), pp. 262-267.
- [13] V. Cruceanu, P.M. Gadea, J. M. Masqué, Para-Hermitian and Para- Kähler Manifolds, Supported by the commission of the European Communities' Action for Cooperation in Sciences and Technology with Central Eastern European Countries n. ERB3510PL920841.

- [14] R. M. Kiehn, Holonomic and Anholonomic Constraints and Coordinates, Frobenius Integrability and Torsion of Various Types, Emeritus, Phys Dept., Univ. Houston, 2001.